JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **52**, 105–119 (1975)

A Mixed Initial Boundary-Value Problem Arising in Neurophysiology

C. V. Pao

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607

Submitted by W. F. Ames

1. Introduction

In the study of biological propagation process in an animal nerve axon, Nagumo *et al.* [7] proposed a model which simulates the active pulse transmission line in an animal nerve axon. The equation of propagation in their model is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^2\, \partial t} - c_1(1 - u + c_2 u^2)\frac{\partial u}{\partial t} - u \qquad (t > 0, x > 0) \tag{1.1}$$

and the boundary and initial conditions are

$$u_t(t, 0) = h(t) \qquad (t \geqslant 0) \tag{1.2}$$

$$u(0, x) = \phi(x), \qquad u_t(0, x) = \psi(x) \qquad (x > 0), \tag{1.3}$$

where $u_t \equiv \partial u/\partial t$, c_1, c_2 are nonnegative constants and h, ϕ, ψ are prescribed functions. The above model was extended by Arima and Hasegawa [1] and Yamaguti [12] to the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^2\, \partial t} - f(u)\frac{\partial u}{\partial t} - g(u) \qquad (t > 0, x > 0), \tag{1.4}$$

where f, g are nonlinear functions of u subjecting various restrictions. However, the propagation medium considered in [1] and [12] is essentially limited to the semiinfinite interval $[0, \infty)$ and the conditions imposed on f, g are somewhat too restrictive. In this paper, we extend the above model to a multidimensional medium and in the meantime weaken the conditions on f, g. Specifically, we consider the following more general equation:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial^3 u}{\partial x_i\, \partial x_j\, \partial t} + \sum_{i=1}^{n} b_i(t, x)\frac{\partial^2 u}{\partial x_i\, \partial t} - f(t, x, u)\, u_t - g(t, x, u)$$
$$(t \in (0, T], x \in \Omega), \tag{1.5}$$

where Ω is an open domain in the n-dimensional Euclidean space R^n and T is an arbitrary finite number. The spatial domain Ω can be either bounded or unbounded. Denote the boundary of Ω by $\partial\Omega$. Then the boundary and initial conditions to be considered are

$$\begin{gathered} \alpha(t, x) \frac{\partial}{\partial \nu} u_t(t, x) + u_t(t, x) = h(t, x) \qquad (t \in [0, T], x \in \partial\Omega) \\ \lim_{|x| \to \infty} u_t(t, x) = 0 \qquad (t \in [0, T]) \end{gathered} \tag{1.6}$$

$$u(0, x) = \phi(x), \qquad u_t(0, x) = \psi(x) \qquad (x \in \Omega), \tag{1.7}$$

where ν is the outward unit normal vector on the boundary $\partial\Omega$ and α is a nonnegative function on $[0, T] \times \partial\Omega$. When $\alpha \equiv 0$ the problem (1.5)–(1.7) becomes a first initial boundary-value problem. Notice that if Ω is bounded we need only the first condition in (1.6) and if $\Omega = R^n$, only the second condition is required. The general consideration includes many special cases such as a half-space, an infinite cylinder, the exterior of a bounded domain, etc. In particular, if $\alpha = 0$ and $\Omega = (0, \infty)$ the first equation in (1.6) is reduced to the form in (1.2). On the other hand, if $\Omega = R^1$ and $a_{ij} = a_{11} = 1$, $b_i = b_1 = 0$, then (1.5) can be reduced to the form of the "nerve axon equation" considered by Evans (cf. [2, 3]) for the Hodgkin–Huxley model (cf. [5]). In this situation, the problem (1.5)–(1.7) is essentially an initial-value problem. For the convenience of later development we sometimes write the two equations in (1.6) by $B[u_t] = h(t, x)$.

Throughout the paper we assume that the coefficients a_{ij}, b_i of the operator

$$L = \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i(t, x) \frac{\partial}{\partial x_i}$$

are continuous on $D \equiv [0, T] \times \bar{\Omega}$, the matrix a_{ij} is positive definite on D (so that L is uniformly elliptic), the functions f, g are continuous on $D \times R^1$, α, h are continuous on $[0, T] \times \partial\Omega$, and ϕ, ψ are continuous on $\bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω. We also assume that the boundary surface $\partial\Omega$ is sufficiently smooth and ψ satisfies the boundary condition (1.6) at $t = 0$ (see also Remark 2.1). Our purpose is to show the existence of a unique solution to the problem (1.5)–(1.7) and to present a constructive method for the determination of the solution. To this goal, we require that f, g satisfy the following local Lipschitz condition:

(H_1) There exist positive constants r, K such that

$$\begin{aligned} |f(t, x, \eta) - f(t, x, \eta')| &\leqslant K |\eta - \eta'|, \\ |g(t, x, \eta) - g(t, x, \eta')| &\leqslant K |\eta - \eta'|, \end{aligned} \qquad ((t, x) \in D, \eta, \eta' \in [-r, r]). \tag{1.8}$$

It will be shown in the following section that if (H_1) holds then the problem (1.5)–(1.7) has a unique "local solution" $u(t, x)$ which can be continued for as long as both $|u|$ and $|u_t|$ remain bounded by r (cf. Theorem 2.3). If, in addition,

(H_2) There exist positive constants M, β such that

$$f(t, x, \eta) \geqslant \beta \, | g(t, x, \eta)| \qquad \text{for } |\eta| > M, \tag{1.9}$$

then the problem (1.5)–(1.7) has a unique global solution. Both local and global solutions can be constructed by successive approximations. The above conditions on f, g are considerably weaker than those required in [1] and [12]. Furthermore, since the conditions (1.8) and (1.9) are obviously satisfied by the functions that appear in (1.1) we see that these results are directly applicable to the problem (1.1)–(1.3). In fact, we will give an explicit recursion formula for the determination of the solution by an iteration process which involves only straightforward integration.

2. Successive Approximations

In this section, we prove the existence of a solution for a so-called "modified problem" of (1.5)–(1.7) by the method of successive approximation. This result leads immediately to the existence of a local solution to the original problem (1.5)–(1.7) and will lead eventually to the existence of a global solution.

Define continuous functions $\tilde{f}$, $\tilde{g}$ in such a way that $\tilde{f}$, $\tilde{g}$ satisfy a global Lipschitz condition in η_1, η_2 while $\tilde{f}(t, x, \eta_1, \eta_2)$, $\tilde{g}(t, x, \eta_1)$ coincide with $f(t, x, \eta_1)\,\eta_2$ and $g(t, x, \eta_1)$, respectively, when $\eta_1, \eta_2 \in [-r, r]$, where $r > 0$ is the constant given in hypothesis (H_1). For example, we may define

$$\tilde{g}(t, x, \eta_1) = \begin{cases} g(t, x, r) & \text{if} \quad \eta_1 \geqslant r \\ g(t, x, \eta_1) & \text{if} \quad |\eta_1| \leqslant r \\ g(t, x, -r) & \text{if} \quad \eta_1 \leqslant -r \end{cases} \tag{2.1}$$

and a similar expression for $\tilde{f}(t, x, \eta_1, \eta_2)$. (For a more specific example for $\tilde{f}$, see [9].) With this modification, the function $\tilde{F}$ defined by

$$\tilde{F}(t, x, \eta_1, \eta_2) = -[\tilde{f}(t, x, \eta_1, \eta_2) + \tilde{g}(t, x, \eta_1)]$$

satisfies the global Lipschitz condition

$$|\tilde{F}(t, x, \eta_1, \eta_2) - \tilde{F}(t, x, \eta_1', \eta_2')| \leqslant K(|\eta_1 - \eta_1'| + |\eta_2 - \eta_2'|) \qquad ((t, x) \in D, \eta_1, \eta_2 \in R^1) \tag{2.2}$$

whenever the functions f, g satisfy the local Lipschitz condition (1.8).

Let $u = u_1$, $u_t = u_2$ and consider the initial boundary-value problem:

$$\begin{aligned} (u_1)_t &= u_2 \\ (u_2)_t - Lu_2 &= \tilde{F}(t, x, u_1, u_2) \end{aligned} \qquad (t \in (0, T], x \in \Omega) \tag{2.3}$$

$$B[u_2] = h(t, x) \qquad (t \in [0, T], x \in \partial\Omega) \tag{2.4}$$

$$u_1(0, x) = \phi(x), \qquad u_2(0, x) = \psi(x) \qquad (x \in \Omega). \tag{2.5}$$

The system (2.3)–(2.5), called a "modified problem" of (1.5)–(1.7), is equivalent to the system (1.5)–(1.7) except with $-[f(t, x, u_1) u_2 + g(t, x, u_1)]$ replaced by $\tilde{F}(t, x, u_1, u_2)$. Since these two functions coincide when $|u_1|$, $|u_2|$ are bounded by r we see that any solution (u_1, u_2) of the problem (2.3)–(2.5) is also a solution of the problem (1.5)–(1.7) with $u = u_1$, $u_t = u_2$ so long as $|u_1|$, $|u_2|$ are bounded by r. To show the existence of a solution to the modified problem we make the transformation $u_1 \to e^{\lambda t} u_1$, $u_2 \to e^{\lambda t} u_2$ and transform the problem (2.3)–(2.5) to the form

$$\begin{aligned} (u_1)_t + \lambda u_1 &= u_2 \\ (u_2)_t - (L - \lambda) u_2 &= F_\lambda(t, x, u_1, u_2) \end{aligned} \qquad (t \in (0, T], x \in \Omega) \tag{2.6}$$

$$B[u_2] = e^{-\lambda t} h(t, x) \qquad (t \in [0, T], x \in \partial\Omega) \tag{2.7}$$

$$u_1(0, x) = \phi(x), \qquad u_2(0, x) = \psi(x) \qquad (x \in \Omega), \tag{2.8}$$

where $\lambda \geqslant 0$ is a constant to be chosen (cf. Theorem 2.1) and

$$F_\lambda(t, x, u_1, u_2) = e^{-\lambda t} \tilde{F}(t, x, e^{\lambda t} u_1, e^{\lambda t} u_2). \tag{2.9}$$

In view of the condition (2.2), the function F_λ also satisfies a global Lipschitz condition in η_1, η_2 and with the same Lipschitz constant K which is independent of λ. For definiteness, we choose $\lambda > K + 1$. Thus to show the existence of a solution to (2.3)–(2.5) it suffices to show the same for the problem (2.6)–(2.8).

Let $C(D)$ be the Banach space of all bounded continuous functions $u(t, x)$ on $D \equiv [0, T] \times \bar{\Omega}$. The norm in $C(D)$ is given by

$$\| u \| = \sup\{| u(t, x)|; (t, x) \in D\}.$$

For simplicity, we assume the following.

(H_3) For some closed subset S of $C(D)$ the functions $f(t, x, u_1) u_2$, $g(t, x, u_1)$ are in S whenever u_1, u_2 are in S and for each $p \in S$ the linear problem

$$\begin{aligned} (u_2)_t - (L - \lambda) u_2 &= p(t, x) && (t \in (0, T], x \in \Omega) \\ B[u_2] &= e^{-\lambda t} h(t, x) && (t \in [0, T], x \in \partial\Omega) \\ u_2(0, x) &= \psi(x) && (x \in \Omega) \end{aligned} \tag{2.10}$$

has a solution in S (see Remark 2.1).

The above assumption ensures that for any $u_1^{(0)}$, $u_2^{(0)}$ in S we can construct a sequence $\{u_1^{(k)}, u_2^{(k)}\}$ successively from the system

$$\begin{aligned} (u_1^{(k)})_t + \lambda u_1^{(k)} &= u_2^{(k-1)}, \qquad u_1(0, x) = \phi(x) \\ (u_2^{(k)})_t - (L - \lambda)\, u_2^{(k)} &= F_\lambda(t, x, u_1^{(k-1)}, u_2^{(k-1)}) \qquad k = 1, 2, \ldots \\ B[u_2^{(k)}] &= e^{-\lambda t} h(t, x), \qquad u_2^{(k)}(0, x) = \psi(x). \end{aligned} \tag{2.11}$$

The construction of $\{u_1^{(k)}, u_2^{(k)}\}$ is clear since the above system consists of two uncoupled (but interrelated) linear systems. We show that if $\tilde{F}$ satisfies the condition (2.2), then the sequence $\{u_1^{(k)}, u_2^{(k)}\}$ converges uniformly to a unique solution of (2.6)–(2.8). To achieve this, we formulate the problem (2.6)–(2.8) as an operator equation.

Let $\mathscr{C}(D) = C(D) \times C(D)$ be the product space with the norm

$$\| U \| = \| u_1 \| + \| u_2 \| \qquad (U = (u_1, u_2) \in \mathscr{C}(D)).$$

Since it seems there is no confusion we use the same norm notation for both $C(D)$ and $\mathscr{C}(D)$. Define operators A_1, A_2, F_λ by

$$\begin{aligned} A_1 u_1 &= (u_1)_t + \lambda u_1, && (u_1 \in D(A_1)), \\ A_2 u_2 &= (u_2)_t - (L - \lambda)\, u_2, && (u_2 \in D(A_2)), \\ (F_\lambda(u_1, u_2))\,(t, x) &= F_\lambda(t, x, u_1(t, x), u_2(t, x)), && (u_1, u_2 \in C(D)), \end{aligned} \tag{2.12}$$

where $D(A_i)$ is the domain of A_i $(i = 1, 2)$ given by

$$\begin{aligned} D(A_1) &= \{u_1 \in C(D);\, (u_1)_t \in C(D),\, u_1(0, x) = \phi(x)\} \\ D(A_2) &= \{u_2 \in C(D);\, (u_2)_t, Lu_2 \in C(D);\, B[u_2] = e^{-\lambda t} h(t, x),\, u_2(0, x) = \psi(x)\}. \end{aligned} \tag{2.13}$$

We next define $\mathscr{A}$, $\mathscr{F}$ by:

$$\begin{aligned} \mathscr{A} U &= (A_1 u_1, A_2 u_2), && (U = (u_1, u_2) \in D(\mathscr{A})), \\ \mathscr{F}(U) &= (u_2, F_\lambda(u_1, u_2)), && (U = (u_1, u_2) \in \mathscr{C}(D)), \end{aligned} \tag{2.14}$$

where $D(\mathscr{A}) = D(A_1) \times D(A_2)$. Then $\mathscr{A}$ is an operator with domain $D(\mathscr{A})$ and range $R(\mathscr{A})$ both in $\mathscr{C}(D)$ while $\mathscr{F}$ maps the whole space $\mathscr{C}(D)$ into itself. With this definition, the problem (2.6)–(2.8) is formulated as an operator equation

$$\mathscr{A} U = \mathscr{F}(U) \qquad (U \in D(\mathscr{A})) \tag{2.15}$$

in the Banach space $\mathscr{C}(D)$. The system (2.11) for the approximations is equivalent to

$$\mathscr{A} U^{(k)} = \mathscr{F}(U^{(k-1)}), \qquad (U^{(k)} \in D(\mathscr{A})). \tag{2.16}$$

The requirement of U and $U^{(k)}$ in $D(\mathscr{A})$ ensures that the solutions of (2.15) and (2.16) satisfy the boundary and initial conditions (2.7) and (2.8). In order to prove the convergence of the sequence $\{U^{(k)}\}$ we need the following lemmas:

LEMMA 2.1. *For each $i = 1, 2$ and any $\lambda > 0$,*

$$\| A_i u - A_i u' \| \geqslant \lambda \| u - u' \| \qquad (u, u' \in D(A_i)). \tag{2.17}$$

Proof. We consider the case for A_2. Since (2.17) is trivially satisfied if $u = u'$ we may assume that $w \equiv u - u' \neq 0$. Moreover, from $\lim w(t, x) = 0$ as $| x | \to \infty$, there exists $(t_0, x_0) \in D$ such that $\| w \| = | w(t_0, x_0)|$. By the definition of $D(A_2)$ we see from $w(0, x_0) = \psi(x_0) - \psi(x_0) = 0$ that $t_0 \in (0, T]$. Furthermore the boundary condition (1.2) implies that $x_0 \in \Omega$. For if x_0 were on $\partial\Omega$ then we would have $w(t_0, x_0) = 0$ if $\alpha(t_0, x_0) = 0$ and

$$(\partial w/\partial \nu)(t_0, x_0) = -w(t_0, x_0)/\alpha(t_0, x_0)$$

if $\alpha(t_0, x_0) \neq 0$. In the latter case, $(\partial w/\partial \nu)(t_0, x_0) \gtrless 0$ according to $w(t_0, x_0) \lesseqgtr 0$ which contradicts the positive maximum (or negative minimum) property of $w(t_0, x_0)$. Knowing $t_0 \in (0, T]$, $x_0 \in \Omega$ we have

$$w(t_0, x_0)\, w_t(t_0, x_0) \geqslant 0, \qquad w(t_0, x_0)\, [(Lw)(t_0, x_0)] \leqslant 0 \tag{2.18}$$

(cf. [4, p. 34; or 8]). It follows from the definition of A_2 that

$$\begin{aligned} w(t_0, x_0)\, [(A_2 u - A_2 u')(t_0, x_0)] &= w(t_0, x_0)\, [w_t(t_0, x_0) - (L - \lambda)\, w(t_0, x_0)] \\ &\geqslant \lambda\, | w(t_0, x_0)|^2. \end{aligned} \tag{2.19}$$

The above relation leads to

$$\| w \|\, \| A_2 u - A_2 u' \| \geqslant | w(t_0, x_0)\, [(A_2 u - A_2 u')(t_0, x_0)]| \geqslant \lambda \| w \|^2, \tag{2.20}$$

which is equivalent to (2.17) for A_2. It is clear that the above argument also shows the relation (2.17) for A_1.

LEMMA 2.2. *For any $\lambda > 0$ the inverse operator $\mathscr{A}^{-1}$ exists on $R(\mathscr{A})$ and*

$$\| \mathscr{A}^{-1} W - \mathscr{A}^{-1} W' \| \leqslant \lambda^{-1} \| W - W' \| \qquad (W, W' \in R(\mathscr{A})). \tag{2.21}$$

Proof. By Lemma 2.1, we have for any $U = (u_1, u_2)$, $U' = (u_1', u_2')$ in $D(\mathscr{A})$,

$$\begin{aligned} \| \mathscr{A} U - \mathscr{A} U' \| &= \| A_1 u_1 - A_1 u_1' \| + \| A_2 u_2 - A_2 u_2' \| \\ &\geqslant \lambda(\| u_1 - u_1' \| + \| u_2 - u_2' \|) \\ &= \lambda \| U - U' \|. \end{aligned}$$

The existence of $\mathscr{A}^{-1}$ and the inequality (2.21) follows immediately.

We now prove the existence of a solution for the transformed problem (2.6)–(2.8).

THEOREM 2.1. *Assume that the hypothesis* (H_3) *and the condition* (2.2) *are satisfied. Then for any* $\lambda > K + 1$ *and any* $U^{(0)} = (u_1^{(0)}, u_2^{(0)})$ *in* $S \times S$ *the sequence* $\{U^{(k)}\} = \{u_1^{(k)}, u_2^{(k)}\}$ *given by* (2.11) *converges in* $\mathscr{C}(D)$ *to a unique solution* $U = (\tilde{u}_1, \tilde{u}_2)$ *of the problem* (2.6)–(2.8). *Furthermore,*

$$\| U^{(k)} - U \| \leqslant \frac{K+1}{\lambda - K - 1} \left(\frac{K+1}{\lambda} \right)^{k-1} \| U^{(1)} - U^{(0)} \|, \qquad k = 1, 2,.... \tag{2.22}$$

Proof. Let $W = (w_1, w_2)$, $W' = (w_1', w_2')$ in $\mathscr{C}(D)$. By the condition (2.2),

$$| F_\lambda(t, x, w_1, w_2) - F_\lambda(t, x, w_1', w_2')| \leqslant K(| w_1 - w_1' | + | w_2 - w_2' |)$$

for all $(t, x) \in D$. The above inequality implies that

$$\| F_\lambda(w_1, w_2) - F_\lambda(w_1', w_2')\| \leqslant K(\| w_1 - w_1' \| + \| w_2 - w_2' \|)$$

and hence

$$\| \mathscr{F}(W) - \mathscr{F}(W')\| \leqslant (K + 1) \| W - W' \|. \tag{2.23}$$

On the other hand, for $W = (w_1, w_2) \in \mathscr{S} = S \times S$ we can find, in view of hypothesis (H_3) and Lemma 2.1, a unique $U = (u_1, u_2)$ such that $A_1 u_1 = w_1$ and $A_2 u_2 = F_\lambda(w_1, w_2)$, that is, $\mathscr{A} U = \mathscr{F}(W)$. Furthermore, by Lemma 2.2 we may write $U = \mathscr{A}^{-1}\mathscr{F}(W)$. It follows from the conditions (2.21) and (2.23) that

$$\| \mathscr{A}^{-1}\mathscr{F}(W) - \mathscr{A}^{-1}\mathscr{F}(W')\| \leqslant \lambda^{-1}(K + 1) \| W - W' \|, \qquad (W, W' \in \mathscr{S}). \tag{2.24}$$

Since $\lambda > K + 1$, the above relation shows that the composite operator $\mathscr{A}^{-1}\mathscr{F}$ is a contraction mapping on $\mathscr{S}$. By the contraction property of $\mathscr{A}^{-1}\mathscr{F}$, the sequence $\{U^{(k)}\}$ given by

$$U^{(k)} = \mathscr{A}^{-1}\mathscr{F}(U^{(k-1)}), \qquad k = 1, 2,... \tag{2.25}$$

converges in $\mathscr{C}(D)$ to a unique $U = (\tilde{u}_1, \tilde{u}_2) \in \mathscr{S}$ such that $U = \mathscr{A}^{-1}\mathscr{F}(U)$ and satisfies the estimate (2.22). This implies that $U \in D(\mathscr{A})$ and is the unique solution of (2.15). Since Eq. (2.25) is equivalent to (2.16) which is the operator equation for (2.11) and since Eq. (2.15) is the operator equation for the problem (2.6)–(2.8) we conclude that the sequence $\{u_1^{(k)}, u_2^{(k)}\}$ given by (2.11) converges in $\mathscr{C}(D)$ to a unique solution $(\tilde{u}_1, \tilde{u}_2)$ of the problem (2.6)–(2.8). This completes the proof of the theorem.

Let $w_1^{(k)} = e^{\lambda t} u_1^{(k)}$ and $w_2^{(k)} = e^{\lambda t} u_2^{(k)}$. Then for each $k = 1, 2,\ldots, \{w_1^{(k)}, w_2^{(k)}\}$ satisfies the system (2.11) for $\lambda = 0$. Since the convergence of $\{u_1^{(k)}, u_2^{(k)}\}$ implies the convergence of $\{w_1^{(k)}, w_2^{(k)}\}$ and since the problem (2.6)–(2.8) coincides with the problem (2.3)–(2.5) when $\lambda = 0$ we therefore have the following conclusion.

THEOREM 2.2. *Assume that the hypothesis* (H_3) *and the condition* (2.2) *are satisfied. Then the sequence* $\{u_1^{(k)}, u_2^{(k)}\}$ *given by* (2.11) *with* $\lambda = 0$ *converges in* $\mathscr{C}(D)$ *to a unique solution* $(\tilde{u}_1, \tilde{u}_2)$ *of the modified problem* (2.3)–(2.5).

Remark 2.1. It is seen from the proof of Theorem 2.1 that the existence of a solution to the linear problem (2.10) assumed in (H_3) is to ensure the existence of the sequence $\{u^{(k)}\}$ in S. The closed property of S is used only to guarantee that the limit function $u = \lim u^{(k)}$ as $k \to \infty$ is in S. These requirements can be replaced by assuming that the coefficients of L and the functions f, g are Hölder continuous and the boundary surface $\partial\Omega$ is sufficiently smooth. Specifically, if the coefficients of L and the function p are in $C^{\alpha,\alpha/2}(\bar{\Omega} \times [0, T])$, $\phi, \psi \in C^{\alpha+2}(\bar{\Omega})$, $h \in C^{\alpha+1,(\alpha+1)/2}(\overline{\partial\Omega} \times [0, T])$ and satisfy the compatibility condition, and if $\partial\Omega$ is of class $C^{\alpha+2}$, where C^{α} and $C^{\alpha,l}$ denote the class of Hölder continuous functions (of exponent α in x, l in t) in their respective domains, then the problem (2.10) has a unique solution $u_2 \in C^{\alpha+2,\alpha/2+1}(\bar{\Omega} \times [0, T])$ (cf. [6, p. 320]). Hence if we assume that $f(t, x, \eta)$, $g(t, x, \eta)$ are in $C^{\alpha,\alpha/2}(\bar{\Omega} \times [0, T])$ for each $\eta \in (-\infty, \infty)$, then by the Lipschitz condition (2.2), $\tilde{F}(t, x, u_1, u_2)$ are in $C^{\alpha,\alpha/2}(\bar{\Omega} \times [0, T])$ whenever u_1, u_2 are in this space. This implies that if $u_1^{(0)}, u_2^{(0)} \in C^{\alpha,\alpha/2}(\bar{\Omega} \times [0, T])$ then the functions $(u_1^{(k)}, u_2^{(k)})$ given by (2.11) are well-defined and are in $C^{\alpha,\alpha/2}(\bar{\Omega} \times [0, T])$ for every $k = 1, 2,\ldots$. The proof of Theorem 2.1 shows that $\{U^{(k)}\} = \{u_1^{(k)}, u_2^{(k)}\}$ converges uniformly to a unique function $U = (\tilde{u}_1, \tilde{u}_2) \in C(D)$. Furthermore, by the continuity of $\mathscr{F}$,

$$\lim \mathscr{A} U^{(k)} = \lim \mathscr{F}(U^{(k-1)}) = \mathscr{F}(U) \qquad \text{as} \qquad k \to \infty.$$

It is obvious from the uniform convergence of $\{u_1^{(k)}, u_2^{(k)}\}$ that $(\tilde{u}_1, \tilde{u}_2)$ satisfies the condition (2.7) and (2.8) and the first equation in (2.6). By a regularity argument it can be shown that $(\tilde{u}_2)_t$ and $L\tilde{u}_2$ exist at every point $(t, x) \in (0, T] \times \Omega$ (e.g., see [4]). Hence the function $U = (\tilde{u}_1, \tilde{u}_2)$ is indeed the desired solution of (2.6)–(2.8) (see [8] for additional remarks).

An immediate consequence of Theorem 2.2 is the following local existence theorem for the original problem.

THEOREM 2.3. *Assume that the hypothesis* (H_3) *and the condition* (1.8) *are satisfied. Then for* $|\phi(x)| < r$, $|\psi(x)| < r$ *the problem* (1.5)–(1.7) *has a unique "local solution"* $u(t, x)$ *in the sense that for some* $T_0 > 0$, $u(t, x)$ *satisfies the* *system* (1.5)–(1.7) *for all* $x \in \bar{\Omega}$, $t \in [0, T_0]$. *The value of* T_0 *is determined by the largest interval on which* $|u(t, x)| \leqslant r$ *and* $|u_t(t, x)| \leqslant r$. *If, in addition, the condition* (1.8) *holds for every finite* r, *where* $K = K(r)$ *may depend on* r, *then* $u(t, x)$ *can be continued for as long as* u *and* u_t *remain bounded on* $\bar{\Omega}$.

Proof. Let $\tilde{u}_1$, $\tilde{u}_2$ be the solution of the modified problem (2.3)–(2.5). Then from $|\tilde{u}_1(0, x)| = |\phi(x)| < r$, $|\tilde{u}_2(0, x)| = |\psi(x)| < r$ and the continuity of $\tilde{u}_1$, $\tilde{u}_2$ there exists $T_0 > 0$ such that $|\tilde{u}_1(t, x)| \leqslant r$, $|\tilde{u}_2(t, x)| \leqslant r$ for $t \in [0, T_0]$, $x \in \bar{\Omega}$. Since $\tilde{F}(t, x, \tilde{u}_1, \tilde{u}_2)$ coincides with

$$-[f(t, x, \tilde{u}_1)\, \tilde{u}_2 + g(t, x, \tilde{u}_1)]$$

when $|\tilde{u}_1| \leqslant r$, $|\tilde{u}_2| \leqslant r$ we see that the function $u(t, x) = \tilde{u}_1(t, x)$ (with $u_t(t, x) = \tilde{u}_2(t, x)$) is the unique solution of (1.5)–(1.7) for $t \in [0, T_0]$, $x \in \bar{\Omega}$. Obviously, the value of T_0 is determined by the largest interval on which $|\tilde{u}_1(t, x)| \leqslant r$, $|\tilde{u}_2(t, x)| \leqslant r$. In case the condition (1.8) holds for every finite r then by continuation (or by taking r sufficiently large) the solution $(\tilde{u}_1, \tilde{u}_2)$ of the problem (2.3)–(2.5) is the unique solution of the problem (1.5)–(1.7) for as long as both $\tilde{u}_1$, $\tilde{u}_2$ remain bounded. This proves the theorem.

Remark 2.2. The proofs of the above theorems show that if the function $f(t, x, u)\, u_t + g(t, x, u)$ in (1.5) is replaced by a more general function $F(t, x, u, u_t)$, which satisfies a local Lipschitz condition in u and u_t, then the results in Theorems 2.2 and 2.3 remain valid. In particular, if F is a polynomial function of u and u_t, that is, if F is of the form

$$F(t, x, u, u_t) = \sum_{i=0}^{n} d_i(t, x)\, u^i u_t^{n-i}, \tag{2.26}$$

where d_i are continuous on D, then since F satisfies the condition (1.8) for every finite r, Theorem 2.3 ensures that the problem (1.5)–(1.7) (with the function F given by (2.26)) has a unique local solution $u(t, x)$ which can be continued for as long as u and u_t remain bounded on $\bar{\Omega}$.

3. Existence of Global Solution

The results of Theorem 2.3 are in analogy to the classical theory of Cauchy problem for ordinary differential equations. As is known from the theory of ordinary differential equations, local Lipschitz condition alone is not sufficient to ensure the existence of a global solution. In order to obtain a solution for the problem (1.5)–(1.7) for all $t \in [0, T]$ it seems necessary to impose some additional conditions on f, g. In this section, we show that if f, g satisfy the condition (1.9) then the solution of the modified problem (2.3)–(2.5) is, in fact, the unique solution of the original problem. Before proving the existence theorem we study some properties of a solution of the problem (1.5)–(1.7) when it exists. For this purpose, we need the following lemma.

LEMMA 3.1. *Let $w(t, x)$ be a continuous function such that $w_t(t, x)$ exists at the point (t_0, x_0). Then the right-derivative of $|w(t, x_0)|$ exists at $t = t_0$ and*

$$|w(t_0, x_0)| \frac{d^+}{dt} (|w(t, x_0)|)_{t=t_0} = w(t_0, x_0)\, w_t(t_0, x_0). \tag{3.1}$$

Proof. A proof of the above lemma has essentially been given in [9] and we sketch it as follows: From the relation

$$\delta^{-1} |[|w(t_0 + \delta, x_0)| - |w(t_0, x_0)|] - [|w(t_0, x_0) + \delta w_t(t_0, x_0)| - |w(t_0, x_0)|]|$$
$$\leqslant \delta^{-1} |w(t_0 + \delta, x_0) - w(t_0, x_0) - \delta w_t(t_0, x_0)|$$

for every $\delta > 0$ we obtain

$$\frac{d^+}{dt} (|w(t, x_0)|)_{t=t_0} = \lim_{\delta \to 0} \delta^{-1}[|w(t_0, x_0) + \delta w_t(t_0, x_0)| - |w(t_0, x_0)|], \tag{3.2}$$

provided the limit at the rightside of (3.2) exists. Without any loss, we may assume that $w(t_0, x_0) \neq 0$. Since for sufficiently small $\delta > 0$,

$$w(t_0, x_0) + \delta w_t(t_0, x_0) \gtrless 0 \qquad \text{according to} \qquad w(t_0, x_0) \gtrless 0$$

we see that

$$|w(t_0, x_0) + \delta w_t(t_0, x_0)| - |w(t_0, x_0)| = (\operatorname{sgn} w(t_0, x_0))\, [\delta w_t(t_0, x_0)]. \tag{3.3}$$

It follows from (3.3) that the limit in (3.2) exists and is equal to $(w(t_0, x_0)/|w(t_0, x_0)|)\, w_t(t_0, x_0)$, which leads to the relation (3.1).

Let $C(\bar{\Omega})$ be the space of all bounded continuous functions $u(x)$ on $\bar{\Omega}$. The norm of $u \in C(\bar{\Omega})$ is defined by

$$\| u \|_\Omega = \sup\{|u(x)|;\, x \in \bar{\Omega}\}.$$

For any bounded continuous function $w(t, x, \eta)$ on $[0, T] \times \bar{\Omega} \times [0, M]$ we set

$$\gamma_w = \sup\{|w(t, x, \eta)|;\, (t, x, \eta) \in [0, T] \times \bar{\Omega} \times [0, M]\}, \tag{3.4}$$

where M is the constant given in hypothesis (H_2). The following theorem gives some bounded property for any solution of the problem (1.5)–(1.7).

THEOREM 3.1. *Let the condition* (1.9) *be satisfied and let* $u(t, x)$ *be a solution of the problem* (1.5)–(1.7) *for* $h(t, x) \equiv 0$. *Then*

$$\| u(t)\|_\Omega \leqslant \| \phi \|_\Omega + M_0 t, \qquad \| u_t(t)\|_\Omega \leqslant M_0\,, \qquad t \in [0, T], \tag{3.5}$$

where

$$M_0 = \max\{\beta^{-1}, e^{\gamma_f T}(\| \psi \|_\Omega + (\gamma_g/\gamma_f))\}. \tag{3.6}$$

Proof. Let $u_1 = u$, $u_2 = u_t$. We show that $\| u_2(t)\|_\Omega \leqslant M_0$ for all $t \in [0, T]$. For each fixed $t_0 \in (0, T)$, let $x_0 \in \bar{\Omega}$ such that $\| u_2(t_0)\|_\Omega = | u_2(t_0, x_0)|$. By the maximum property of $| u(t_0, x_0)|$ on $\bar{\Omega}$ it is easily seen from the proof of Lemma 2.1 (with $h(t, x) \equiv 0$) that $x_0 \in \Omega$ and $u_2(t_0, x_0)\,[(Lu_2)\,(t_0, x_0)] \leqslant 0$. Since $u(t_0, x_0)$ satisfies Eq. (1.5), an application of Lemma 3.1 leads to

$$\begin{aligned} | u_2(t_0, x_0)| \frac{d^+}{dt} (| u_2(t, x_0)|)_{t=t_0} &= u_2(t_0, x_0)\,[(Lu_2)\,(t_0, x_0) - f(t_0, x_0, u_1(t_0, x_0))\, u_2(t_0, x_0) \\ &\quad - g(t_0, x_0, u_1(t_0, x_0))] \\ &\leqslant - f(t_0, x_0, u_1(t_0, x_0))\, | u_2(t_0, x_0)|^2 \\ &\quad + | u_2(t_0, x_0)|\, | g(t_0, x_0, u_1(t_0, x_0))|\,. \end{aligned} \tag{3.7}$$

Hence if $| u_2(t_0, x_0)| \neq 0$ we can divide (3.7) by $| u_2(t_0, x_0)|$ to obtain

$$\begin{aligned} &\frac{d^+}{dt} (| u_2(t, x_0)|)_{t=t_0} \\ &\quad \leqslant - f(t_0, x_0, u_1(t_0, x_0))\, | u_2(t_0, x_0)| + | g(t_0, x_0, u_1(t_0, x_0))|\,. \end{aligned} \tag{3.8}$$

The above inequality holds for any $t_0 \in (0, T)$ and a corresponding point $x_0 \in \bar{\Omega}$ whenever $\| u(t_0)\|_\Omega \neq 0$. Consider the case where $| u_1(t_0, x_0)| > M$. Then if $| u_2(t_0, x_0)| \geqslant \beta^{-1}$ the condition (1.9) implies that the rightside of (3.8) is nonpositive and thus $| u_2(t, x_0)|$ is nonincreasing at $t = t_0$. Since this is true for any $t_0 \in (0, T)$ we see by starting from a sufficiently small $t_0 > 0$ that $\| u_2(t_0)\|_\Omega \leqslant \| \psi \|_\Omega$ when $\| \psi \|_\Omega > \beta^{-1}$ and $\| u_2(t_0)\|_\Omega \leqslant \beta^{-1}$ when $\| \psi \|_\Omega \leqslant \beta^{-1}$. This relation holds for as long as $| u_1(t_0, x_0)| > M$. On the other hand, when $| u_1(t_0, x_0)| \leqslant M$ the inequality (3.8) implies that

$$\frac{d^+}{dt} (| u_2(t, x_0)|)_{t=t_0} \leqslant \gamma_f\, | u_2(t_0, x_0)| + \gamma_g\,,$$

where γ_f, γ_g are defined by (3.4). Since the usual rules of differentiation hold for right-derivative of a continuous function the above inequality can be put in the form

$$\frac{d^+}{dt} [e^{-\gamma_f t}(| u_2(t, x_0)| + (\gamma_g/\gamma_f))]_{t=t_0} \leqslant 0. \tag{3.9}$$

Hence the function $e^{-\gamma_f t}(|u_2(t, x_0)| + (\gamma_g/\gamma_f))$ is nonincreasing at $t = t_0$. As in the previous case, we conclude that

$$e^{-\gamma_f t_0}(\| u_2(t_0)\|_\Omega + (\gamma_g/\gamma_f)) \leqslant \| \psi \|_\Omega + (\gamma_g/\gamma_f), \tag{3.10}$$

so long as $|u_1(t_0, x_0)| \leqslant M$. Since at any $t_0 \in (0, T)$ (and a corresponding $x_0 \in \bar{\Omega}$) either $|u(t_0, x_0)| > M$ or $|u(t_0, x_0)| \leqslant M$, we obtain from the above conclusion that either $\| u_2(t_0)\|_\Omega$ is bounded by β^{-1} or by $e^{\gamma_f t_0}(\| \psi \|_\Omega + \gamma_g/\gamma_f)$. In any case we have $\| u_2(t_0)\|_\Omega \leqslant M_0$. Finally, the boundedness of $\| u_1(t_0)\|_\Omega$ follows immediately from $\| u_2(t_0)\|_\Omega \leqslant M_0$. This proves the theorem.

When $h(t, x) \neq 0$ the solution u and u_t are also bounded on D. To see this we consider the case $\alpha(t, x) = 0$ for the sake of simplicity. Notice that in this case the first boundary condition in (1.6) is replaced by $u_2(t, x) = h(t, x)$. The following additional notation will be used:

$$\bar{h} = \sup\{|h(t, x)|; t \in [0, T], x \in \partial\Omega\}. \tag{3.11}$$

THEOREM 3.2. *Let the condition* (1.9) *be satisfied and let $u(t, x)$ be a solution of* (1.5)–(1.7) *for $\alpha(t, x) \equiv 0$. Then*

$$\| u(t)\|_\Omega \leqslant \| \phi \|_\Omega + M_h t, \qquad \| u_t(t)\|_\Omega \leqslant M_h, \qquad (t \in [0, T]), \tag{3.12}$$

where

$$M_h = \max\{M_0, \bar{h}\}.$$

Proof. Let $u_1 = u$, $u_2 = u_t$ and let $\| u_2(t_0)\|_\Omega = |u_2(t_0, x_0)|$, where $t_0 \in (0, T)$ is fixed. By the same argument as in the proof of Theorem 3.1 we see that if $x_0 \in \Omega$ then the relations in (3.12) hold with $M_h = M_0$. In case $x_0 \in \partial\Omega$ the boundary condition implies that $|u_2(t_0, x_0)| = |h(t_0, x_0)| \leqslant \bar{h}$. In either case we have $\| u_2(t_0)\|_\Omega \leqslant M_h$. The boundedness of $\| u_1(t)\|_\Omega$ follows immediately.

With the results of Theorems 3.1 and 3.2 we now show the existence of a global solution to the problem (1.5)–(1.7).

THEOREM 3.3. *Let the hypotheses* (H_1)–(H_3) *be satisfied for some*

$$r \geqslant r_0 \equiv \max\{\| \phi \|_\Omega + M_h T, M_h\}$$

and let either $h(t, x) \equiv 0$ or $\alpha(t, x) \equiv 0$. Define modifications $\tilde{f}$, $\tilde{g}$ as in (2.1) *with $r = r_0$. Then the problem* (1.5)–(1.7) *has a unique solution $u(t, x)$ which satisfies the relation* (3.5) *when $h(t, x) \equiv 0$ and the relation* (3.12) *when $\alpha(t, x) \equiv 0$. Furthermore, the pair (u, u_t) is the limit of the sequence $\{u_1^{(k)}, u_2^{(k)}\}$ determined from the problem* (2.11) *with $\lambda = 0$.*

Proof. By the definition of the modifications $\tilde{f}$, $\tilde{g}$ and the hypothesis (H_1), the function $\tilde{F} \equiv -(\tilde{f} + \tilde{g})$ satisfies the global Lipschitz condition (2.2). Hence by Theorem 2.2, the sequence $\{u_1^{(k)}, u_2^{(k)}\}$ given by the problem (2.11) (with $\lambda = 0$) converges to a unique solution $(\tilde{u}_1, \tilde{u}_2)$ of the modified problem (2.3)–(2.5). Since $\tilde{f}$, $\tilde{g}$ also satisfy the condition (1.9) when it is satisfied by f, g we obtain from Theorems 3.1 and 3.2 that the relations (3.5) or (3.12) hold for $(\tilde{u}_1, \tilde{u}_2)$ according to $h(t, x) = 0$ or $\alpha(t, x) = 0$. This implies, in view of the choice of r, that $\tilde{f}(t, x, \tilde{u}_1, \tilde{u}_2)$ and $g(t, x, \tilde{u}_1)$ coincide with $f(t, x, \tilde{u}_1)\, \tilde{u}_2$ and $g(t, x, \tilde{u}_1)$, respectively, for all $(t, x) \in D$. It follows from the equivalence between the problems (1.5)–(1.7) and (2.3)–(2.5) that $u(t, x) \equiv \tilde{u}_1(t, x)$ (with $u_t = \tilde{u}_2$) is the unique solution of the original problem (1.5)–(1.7). This also shows that the sequence $\{u_1^{(k)}, u_2^{(k)}\}$ converges to (u, u_t). This proves the theorem.

In the special case of the problem (1.1)–(1.3), where

$$f(t, x, u) = c_1(1 - u + c_2 u^2), \qquad g(u) = u, \tag{3.13}$$

the hypotheses (H_1), (H_2) are fulfilled. In fact, the condition (1.9) holds with, for example, $\beta = c_1$, $M = 2/c_2$. By Remark 2.1, we have the following conclusion:

COROLLARY. *The problem* (1.1)–(1.3) *has a unique solution which satisfies the condition* (3.12). *The constant* M_0 *in* (3.12) *can be obtained from* (3.4) *and* (3.6) *with* $\beta = c_1$, $M = 2/c_2$.

4. CONSTRUCTION OF SOLUTION

When the Green's function of the linear problem (2.10) is constructed we can obtain an explicit recursion formula for the calculation of the approximations for the problem (1.5)–(1.7). In the following we give such a formula for the special problem (1.1)–(1.3) as an illustration. The same approach can be used for more general operator L and higher dimensional spatial domains.

Consider the linear problem

$$\begin{gathered} (u_2)_t = (u_2)_{xx} + p(t, x), \qquad (t \in (0, T], x \in \Omega), \\ u_2(t, 0) = h(t), \qquad \lim_{x \to \infty} u_2(t, x) = 0, \qquad (t \in [0, T]), \\ u_2(0, x) = \psi(x), \qquad (x \in \Omega), \end{gathered} \tag{4.1}$$

where $\Omega = (0, \infty)$. Assume that p is Hölder continuous in D and $|\, p\, |$ is of order $O(e^{\beta x^2})$ as $x \to \infty$, where $\beta > 0$ is a constant. Then the solution of (4.1) is given by:

$$u_2(t, x) = \int_0^t \int_0^\infty G(t, x \mid \tau, y)\, p(\tau, y)\, dy\, d\tau + \int_0^\infty G(t, x \mid 0, y)\, \psi(y)\, dy + \int_0^t \frac{\partial G}{\partial y}(t, x \mid \tau, 0)\, h(\tau)\, d\tau, \tag{4.2}$$

where G is the Green's function given by

$$G(t, x \mid \tau, y) = \frac{H(t-\tau)}{[4\pi(t-\tau)]^{1/2}} \left[\exp\left(-\frac{|x-y|}{4(t-\tau)}\right) - \exp\left(-\frac{|x+y|}{4(t-\tau)}\right)\right] \tag{4.3}$$

and H is the Heaviside function (e.g., see [11, p. 199]). Let $(u_1^{(0)}, u_2^{(0)})$ be any pair of Hölder continuous functions such that

$$\lim u_1^{(0)}(t, x) = \lim u_2^{(0)}(t, x) = 0$$

as $x \to \infty$. Since $\tilde{f}(t, x, u_1)\, u_2$ and $\tilde{g}(t, x, u_1)$ are Hölder continuous in D and is of order $O(e^{\beta x^2})$ as $x \to \infty$, when u_1, u_2 have these properties we can replace $p(t, x)$ in (4.2) by $\tilde{F}(t, x, u_1^{(k-1)}(t, x), u_2^{(k-1)}(t, x))$ to obtain the sequence $\{u_1^{(k)}, u_2^{(k)}\}$ from the recursion formula:

$$u_1^{(k)}(t, x) = \phi(x) + \int_0^t u_2^{(k-1)}(\tau, x)\, d\tau$$

$$u_2^{(k)}(t, x) = \int_0^t \int_0^\infty G(t, x \mid \tau, y)\, \tilde{F}(\tau, y, u_1^{(k-1)}(\tau, y), u_2^{(k-1)}(\tau, y))\, dy\, d\tau + \int_0^\infty G(t, x \mid 0, y)\, \psi(y)\, dy + \int_0^t \frac{\partial G}{\partial y}(t, x \mid \tau, 0)\, h(\tau)\, d\tau \qquad k = 1, 2, \ldots \tag{4.4}$$

The sequence $\{u_1^{(k)}, u_2^{(k)}\}$ given above is well-defined and satisfies the problem (2.11) for the special case $Lu_2 = (u_2)_{xx}$, $\lambda = 0$, $\alpha(t, x) \equiv 0$. Notice that in the formula (4.4), $\tilde{F}(t, x, u_1, u_2) = -[\tilde{f}(t, x, u_1, u_2) + \tilde{g}(t, x, u_1)]$ and $\tilde{f}$, $\tilde{g}$ are the respective modifications of $f(t, x, u_1)\, u_2$ and $g(t, x, u_1)$. By the application of Theorem 3.3 and its corollary we have the following conclusion:

THEOREM 4.1. *Let $(u_1^{(0)}, u_2^{(0)})$ be any Hölder continuous functions in D such that* $\lim u_1^{(0)}(t, x) = \lim u_2^{(0)}(t, x) = 0$ *as $k \to \infty$. Then the sequence $\{u_1^{(k)}, u_2^{(k)}\}$ given by* (4.4) *converges to a unique pair $(\tilde{u}_1, \tilde{u}_2)$ such that $u \equiv \tilde{u}_1$ (with $u_t \equiv \tilde{u}_2$) is the unique solution of the problem* (1.1)–(1.3).

It is to be noted that in the construction of the solution for the problem (1.1)–(1.3) (and, in general, for the problem (1.5)–(1.7)) the modifications $\tilde{f}$, $\tilde{g}$ should be used in the recursion formula (4.4) since the approximations of the modified problem are not necessarily the same as those of the original problem even though the two systems have the same solution. In fact, it is not known whether or not the sequence of approximations converges if the modifications $\tilde{f}$, $\tilde{g}$ are replaced by the original functions. In using the recursion formula (4.4) for computational purpose, the third integral term in the second equation in (4.4) is not quite satisfactory near $x = 0$. In this situation, a little trick may be necessary to circumvent this difficulty (cf. [11, p. 208]).

References

1. R. Arima and Y. Hasegawa, On global solutions for mixed problem of a semi-linear differential equation, *Proc. Japan Acad.* **39** (1963), 721–725.
2. J. W. Evans and N. Shenk, Solutions to axon equations, *Biophy. J.* **10** (1970), 1090–1101.
3. J. W. Evans, Nerve axon equations I, Linear approximations, *Indiana Univ. Math. J.* **21** (1972), 877–885.
4. A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice-Hall, Englewood Cliffs, New Jersey, 1964.
5. A. L. Hodgkin and A. F. Huxley, A quantitative description of membrane current and its application to conduction and exitation in nerve, *J. Physiol.* **117** (1952), 500–544.
6. O. A. Ladzenskaya, V. A. Solonnikov, and N. N. Uralceva, "Linear and Quasilinear Equations of Parabolic Type," American Mathematical Society, Providence, Rhode Island, 1968.
7. J. Nagumo, S. Arimoto, and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proc. IRE* **50** (1962), 2061–2070.
8. C. V. Pao, Successive approximations of some nonlinear initial boundary-value problems, *SIAM J. Math. Anal.* **5** (1974), 91–102.
9. C. V. Pao, Positive solution of a nonlinear diffusion system arising in chemical reactors, *J. Math. Anal. Appl.* **46** (1974), 820–835.
10. D. H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary-value problems, *Indiana Univ. Math. J.* **21** (1972), 979–1000.
11. I. Stakgold, "Boundary Value Problems of Mathematical Physics," Vol. II, MacMillan, New York, 1968.
12. M. Yamaguti, The asymptotic behavior of the solution of a semi-linear partial differential equation related to an active pulse transmission line, *Proc. Japan Acad.* **39** (1963), 726–730.